

Some calculations in Analysis (Wallis & Stirling formulas)

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1. Wallis formula.

Let $I_n := \int_0^{\pi/2} \sin^n x dx, n \in \mathbb{N} \cup \{0\}$. Follow to definition we have $I_0 = \frac{\pi}{2}$,

$I_1 = \int_0^{\pi/2} \sin x dx = (-\cos x)_0^{\pi/2} = 1$. Note that for any $x \in [0, \pi/2]$ holds

$$0 \leq \sin x \leq 1 \Rightarrow \sin^{n+1} x \leq \sin^n x \leq \sin^{n-1} x \Rightarrow$$

$$(1) \quad I_{n+1} \leq I_n \leq I_{n-1}, n \in \mathbb{N}.$$

Also note that $I_n = \int_0^{\pi/2} \sin^{n-2} x (1 - \cos^2 x) dx = I_{n-2} - \int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx$, where

$$\int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx = \left[\begin{array}{l} u' = \sin^{n-2} x \cos x; u = \frac{\sin^{n-1} x}{n-1} \\ v = \cos x; v' = -\sin x \end{array} \right] =$$

$$\left(\frac{\sin^{n-1} x}{n-1} \cdot \cos x \right)_0^{\pi/2} + \int_0^{\pi/2} \frac{\sin^n x}{n-1} dx = \frac{1}{n-1} I_n. \text{ Hence, } I_n = I_{n-2} - \frac{1}{n-1} I_n \Leftrightarrow$$

$$(2) \quad I_n = \frac{n-1}{n} I_{n-2}, n \geq 2.$$

In particular we have $I_{2n} = \frac{2n-1}{2n} I_{2(n-1)}, n \in \mathbb{N}$.

$$\text{Since } I_{2n} = \frac{2n-1}{2n} I_{2(n-1)} \Leftrightarrow \frac{I_{2n}}{I_{2(n-1)}} = \frac{2n-1}{2n} \Rightarrow \prod_{k=1}^n \frac{I_{2k}}{I_{2(k-1)}} = \prod_{k=1}^n \frac{2k-1}{2k} \Leftrightarrow$$

$$\frac{I_{2n}}{I_0} = \frac{(2n-1)!!}{(2n)!!} \Leftrightarrow I_{2n} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}, n \in \mathbb{N} \cup \{0\}.$$

$$\text{Also, } I_{2n+1} = \frac{2n}{2n+1} I_{2n-1} \Rightarrow \frac{I_{2n+1}}{I_1} = \frac{(2n)!!}{(2n+1)!!} \Leftrightarrow I_{2n+1} = \frac{(2n)!!}{(2n+1)!!}.$$

Then by replacing n in inequality (1) with $2n$ we obtain

$$\frac{(2n)!!}{(2n+1)!!} \leq \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2} \leq \frac{(2n-2)!!}{(2n-1)!!} \Leftrightarrow \frac{((2n)!!)^2}{(2n+1)!!(2n-1)!!} \leq \frac{\pi}{2} \leq \frac{(2n)!!(2n-2)!!}{((2n-1)!!)^2} \Leftrightarrow$$

$$\frac{((2n)!!)^2(2n-1)!!}{(2n+1)!!((2n-1)!!)^2} \leq \frac{\pi}{2} \leq \frac{((2n)!!)^2(2n-2)!!}{((2n-1)!!)^2(2n)!!} \Leftrightarrow$$

$$\frac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)} \leq \frac{\pi}{2} \leq \frac{((2n)!!)^2}{((2n-1)!!)^2 \cdot 2n} \Leftrightarrow$$

$$(3) \quad a_n \leq \frac{\pi}{2} \leq a_n \frac{2n+1}{2n}, \text{ where } a_n := \frac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)} = \prod_{k=1}^n \frac{4k^2}{4k^2-1}.$$

Since $a_n \uparrow \mathbb{N}$ and $a_n \leq \frac{\pi}{2}$ then there is $a := \lim_{n \rightarrow \infty} a_n \leq \frac{\pi}{2}$ and, therefore,

$$\lim_{n \rightarrow \infty} \left(a_n \cdot \frac{2n+1}{2n} \right) = a.$$

Hence $\frac{\pi}{2} = \lim_{n \rightarrow \infty} \prod_{k=1}^n \frac{4k^2}{4k^2-1}$ that is $\frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{((2n)!!)^2}{((2n-1)!!)^2(2n+1)}$ (Wallis Formula).

2. Stirling Formula.

Lemma (extracted from [1])

There is positive constant a such that for any $n \in \mathbb{N}$ holds inequality

$$(1) \quad \left(\frac{n}{e}\right)^n \sqrt{an} < n! < \left(\frac{n}{e}\right)^n \sqrt{an} \cdot e^{\frac{1}{12n}}.$$

Proof.

1. First we will prove inequality $e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}$ for any $n \in \mathbb{N}$.

Note that sequence $\left(\left(1 + \frac{2}{n-1}\right)^n\right)_{n \geq 2}$ is decreasing.

$$\text{Indeed, } \left(1 + \frac{2}{n-1}\right)^n > \left(1 + \frac{2}{n}\right)^{n+1} \Leftrightarrow$$

$$\left(\frac{n+1}{n-1}\right)^n > \left(\frac{n+2}{n}\right)^{n+1} \Leftrightarrow \left(\frac{n(n+1)}{(n-1)(n+2)}\right)^n > 1 + \frac{2}{n} \Leftrightarrow$$

$$\left(1 + \frac{2}{(n-1)(n+2)}\right)^n > 1 + \frac{2}{n}.$$

Applying inequality $(1+a)^n \geq 1+na + \frac{n(n-1)}{2}a^2, a > 0, n \in \mathbb{N}$ for $a = \frac{2}{(n-1)(n+2)}$

$$\text{we obtain } \left(1 + \frac{2}{(n-1)(n+2)}\right)^n \geq 1 + \frac{2n}{(n-1)(n+2)} + \frac{n(n-1)}{2} \cdot \frac{4}{(n-1)^2(n+2)^2} =$$

$$1 + \frac{2n}{(n-1)(n+2)} + \frac{2n}{(n-1)(n+2)^2} \text{ and } 1 + \frac{2n}{(n-1)(n+2)} + \frac{2n}{(n-1)(n+2)^2} > 1 + \frac{2}{n} \Leftrightarrow$$

$$(n+3)n^2 > (n-1)(n+2)^2 \Leftrightarrow n^3 + 3n^2 > n^3 + 3n^2 - 4 \Leftrightarrow 4 > 0. \blacksquare$$

Since $\left(1 + \frac{2}{n-1}\right)^n > \left(1 + \frac{2}{n}\right)^{n+1} > \left(1 + \frac{2}{n+1}\right)^{n+2}$ then

$$(2) \quad \left(1 + \frac{2}{n-1}\right)^n > \left(1 + \frac{2}{n+1}\right)^{n+2}$$

By replacing n in (2) with $2n+1$ we obtain $\left(1 + \frac{1}{n}\right)^{2n+1} > \left(1 + \frac{1}{n+1}\right)^{2n+3} \Leftrightarrow$

$$\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} > \left(1 + \frac{1}{n+1}\right)^{n+1+\frac{1}{2}} \text{ and, since } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} = e \text{ then}$$

$$e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}, n \in \mathbb{N}.$$

2. Consider Taylor Representation for function $\ln \frac{1+x}{1-x}$ for $x \in (0, 1)$:

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(-x)^k}{k} =$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k} + \sum_{k=1}^{\infty} \frac{x^k}{k} = \sum_{k=1}^{\infty} \frac{((-1)^{k-1} + 1)x^k}{k} = \sum_{k=1}^{\infty} \frac{2x^{2k-1}}{2k-1}.$$

Since $\frac{n+1}{n} = \frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}$ then by replacing x in $\ln \frac{1+x}{1-x} = \sum_{k=1}^{\infty} \frac{2x^{2k-1}}{2k-1}$ with $\frac{1}{2n+1}$

we obtain

$$\ln \frac{n+1}{n} = \sum_{k=1}^{\infty} \frac{2}{2k-1} \cdot \frac{1}{(2n+1)^{2k-1}} = \frac{2}{2n+1} \left(1 + \sum_{k=2}^{\infty} \frac{1}{(2k-1)(2n+1)^{2(k-1)}}\right) <$$

$$\frac{2}{2n+1} \left(1 + \sum_{k=1}^{\infty} \frac{1}{3(2n+1)^{2k}}\right) = \frac{2}{2n+1} \left(1 + \frac{1}{3} \cdot \frac{\frac{1}{(2n+1)^2}}{1 - \frac{1}{(2n+1)^2}}\right) =$$

$$\frac{1}{n + \frac{1}{2}} \left(1 + \frac{1}{12n(n+1)} \right) \Rightarrow \left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}.$$

Thus we double inequality $e < \left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$.

$$\text{Since } e < \left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} \Leftrightarrow e < \frac{(n+1)^{n+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \Leftrightarrow e(n+1) < \frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \Leftrightarrow$$

$$\frac{e^{n+1}(n+1)!}{e^n n!} < \frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} \Leftrightarrow \frac{n^{n+\frac{1}{2}}}{e^n n!} < \frac{(n+1)^{(n+1)+\frac{1}{2}}}{e^{n+1}(n+1)!}, n \geq 1 \text{ then sequence } (a_n)_{n \geq 1}$$

where $a_n := \frac{n^{n+\frac{1}{2}}}{e^n n!}$, is increasing. Since $\left(1 + \frac{1}{n} \right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}} \Leftrightarrow$

$$\frac{(n+1)^{n+1+\frac{1}{2}}}{n^{n+\frac{1}{2}}} < \frac{(n+1)!}{n!} \cdot e^{n+1-n+\frac{1}{12n}-\frac{1}{12(n+1)}} \Leftrightarrow \frac{(n+1)^{n+1+\frac{1}{2}}}{(n+1)! e^{n+1-\frac{1}{12(n+1)}}} < \frac{n^{n+\frac{1}{2}}}{n! e^{n-\frac{1}{12n}}}$$

then sequence $(b_n)_{n \geq 1}$, where $b_n := \frac{n^{n+\frac{1}{2}} e^{\frac{1}{12n}}}{n! e^n} = a_n e^{\frac{1}{12n}}$ is decreasing.

Since $e^{-1} = a_1 \leq a_n < b_n \leq b_1 = e^{-\frac{11}{12}}$ and $b_n = a_n e^{\frac{1}{12n}}$ then both sequences converge to the same limit.

$$\text{Let } \frac{1}{a} := \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n \text{ then } e^{\frac{11}{12}} < a < e \text{ and } a_n < \frac{1}{a} < b_n \Leftrightarrow \frac{1}{b_n} < a < \frac{1}{a_n} \Leftrightarrow$$

$$\frac{n! e^n}{n^{n+\frac{1}{2}} e^{\frac{1}{12n}}} < a < \frac{e^n n!}{n^{n+\frac{1}{2}}} \Leftrightarrow \left(\frac{n}{e} \right)^n \sqrt{an} < n! < \left(\frac{n}{e} \right)^n \sqrt{an} \cdot e^{\frac{1}{12n}} \Leftrightarrow$$

$$(3) \quad n! = \left(\frac{n}{e} \right)^n \sqrt{an} \cdot e^{\frac{\theta_n}{12n}}, \text{ where } \theta_n \in (0, 1)$$

Using Wallis formula we can obtain $a = 2\pi$.

Indeed, since by Wallis formula $\frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{((2n)!!)^2}{((2n-1)!!)^2 (2n+1)}$ and

$$\frac{(2n)!!}{(2n-1)!!} = \frac{((2n)!!)^2}{(2n)!} = \frac{2^{2n} (n!)^2}{(2n)!} = \frac{2^{2n} \left(\left(\frac{n}{e} \right)^n \sqrt{an} \cdot e^{\frac{\theta_n}{12n}} \right)^2}{\left(\frac{2n}{e} \right)^{2n} \sqrt{a \cdot 2n} \cdot e^{\frac{\theta_{2n}}{24n}}} =$$

$$\frac{2^{2n} \left(\frac{n}{e} \right)^{2n} a n \cdot e^{\frac{\theta_n}{6n}}}{\left(\frac{2n}{e} \right)^{2n} \sqrt{a \cdot 2n} \cdot e^{\frac{\theta_{2n}}{24n}}} = \frac{\sqrt{an} e^{\frac{\theta_n}{6n} - \frac{\theta_{2n}}{24n}}}{\sqrt{2}}$$

$$\text{then } \frac{\pi}{2} = \lim_{n \rightarrow \infty} \frac{an e^{\frac{\theta_n}{6n} - \frac{\theta_{2n}}{24n}}}{2(2n+1)} = \frac{a}{4} \Leftrightarrow a = 2\pi.$$

$$\text{So, } n! = \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \cdot e^{\frac{\theta_n}{12n}}, \text{ that is } \lim_{n \rightarrow \infty} \frac{n!}{\left(\frac{n}{e} \right)^n \sqrt{2\pi n}} = 1.$$

**1. Mathematical Reflection n.1, 2011, Problem U181,
Solution by Arkady Alt, pp.16-21.**